Critical behavior of the chain-generating function of self-avoiding walks on the Sierpinski gasket family: The Euclidean limit

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We study self-avoiding walks (SAW's) on the generalized Sierpinski gasket family of fractals. Each fractal can be labeled by an integer b $(2 \leqslant b \leqslant \infty)$, so that the fractal and spectral dimensions tend to the Euclidean value 2 when $b \rightarrow \infty$. By using an exact enumeration technique to obtain the series expansion for the chain-generating function of SAW's on these lattices, we calculate the associated critical exponent γ_b for $2 \leqslant b \leqslant 100$. The large-$b$ behavior of γ_b is the first numerical result consistent with the asymptotic convergence toward the Euclidean value γ_E. We also give an analytic argument supporting the assumption that $\lim_{b\rightarrow\infty} \gamma_b \rightarrow \gamma_E$. [S1063-651X(98)14008-4]

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I. INTRODUCTION

Recently, increased attention has been focused on the limiting behavior of the critical properties of statistical systems on fractals when underlying fractal geometrical parameters such as the fractal (D_F) or spectral (D_S) dimensions approach the Euclidean values. This convergence is not ensured, because in order to obtain the critical properties of any statistical system on a fractal family (labeled by b), one should analyze the results in the thermodynamic limit ($N \rightarrow \infty$) for each b, while the critical properties on the limiting Euclidean lattice is obtained when the geometrical limit ($b \rightarrow \infty$) is taken *before* the thermodynamic limit.

In this work we study the critical behavior of self-avoiding walks (SAW's) on a family of regular fractals embedded in the two-dimensional Euclidean space, the generalized Sierpinski gaskets (GSG's). Each member of the fractal family is labeled by an integer $b \geqslant 2$, and can be obtained as the result of an infinite iterative process in which a triangular structure is enlarged b times and a generator is reproduced in $b(b+1)/2$ smallest triangles of the enlarged structure. The generator is the initial structure (see, for instance, Fig. 1 of Ref. [1]). For this fractal family, both D_F and D_S tends to the Euclidean value 2 when $b \rightarrow \infty$.

We present results for $b \leqslant 100$ based on the series expansion method. The series expansion technique gives the most reliable results for the Euclidean lattices. This suggests the importance of extending it for fractal lattices.

We consider the chain-generating function for SAW's on a particular fractal,

$$C_b(x) = \sum_{n=1}^{\infty} c_n(b) x^n, \tag{1}$$

where $c_n(b)$ is the number of distinct n-step SAW's per number of sites of the lattice and x, the fugacity, is the weight factor for each step. Near a critical fugacity x_c,

$$C_b(x) \sim (x - x_c)^{-\gamma_b}, \tag{2}$$

where γ_b is the critical exponent and $\mu_b \equiv (x_c)^{-1}$ is the connective constant.

The critical behavior of SAW's on the GSG family has recently been studied by series expansion method [1]. The critical fugacity μ_b was numerically estimated, and it was found that μ_b tends to the triangular value μ_T when $b \rightarrow \infty$.

However, the asymptotic behavior of critical exponents is still a controversial issue. SAW's were studied on other fractal families which are considered to belong to the same universality class as the generalized Sierpinski gasket family studied here. Numerical results for the critical exponents γ_b and ν_b were obtained via the Monte Carlo renormalization group (MCRG) for $b \leqslant 80$ [2]. Although the range of b was not sufficiently large to allow a numerical estimate of the asymptotic behavior, the results of γ_b and ν_b exhibit a monotonic behavior with b that depart from the respective Euclidean values ($\gamma_E = 1.34$ and $\nu_E = 0.75$) as b increases. On the other hand, finite-size scaling (FSS) arguments [3]

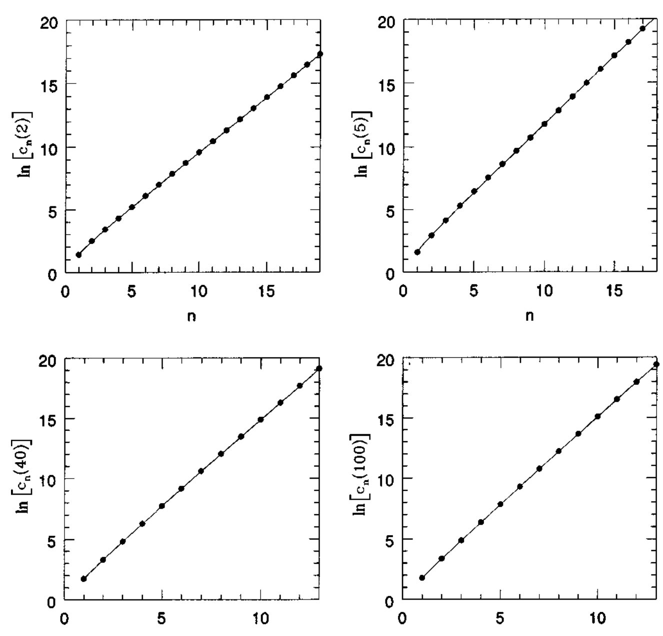

FIG. 1. Plot of $\ln c_n(b)$ vs n for $b = 2$, 5, 40, and 100, respectively. For comparison, we plot the adjusted curve $[\ln A_b + n \ln \mu_b + (\gamma_b - 1) \ln n]$ vs n.

TABLE I. Density of n-step SAW's $c_n(b)$ for the fractal lattices labeled by $b=8$ and 80.

b	n	$c_n(b)$
8	1	7560/1512
8	2	31752/1512
8	3	121632/1512
8	4	449568/1512
8	5	1622670/1512
8	6	5741022/1512
8	7	19885110/1512
8	8	2429613144/54432
8	9	8076704832/54432
8	10	26308558584/54432
8	11	84118535424/54432
8	12	264733395192/54432
8	13	823003590024/54432
8	14	2537259389280/54432
8	15	7787438864352/54432
8	16	23885916609744/54432
80	1	62966160/10750320
80	2	308687760/10750320
80	3	1394494728/10750320
80	4	6137824104/10750320
80	5	26666991150/10750320
80	6	114839951646/10750320
80	7	491059627350/10750320
80	8	2089451131974/10750320
80	9	8855701290984/10750320
80	10	37414379751984/10750320
80	11	157666123368396/10750320
80	12	662982179959008/10750320
80	13	2782741988780340/10750320

suggest that $\lim_{b\to\infty}\nu_b=\nu_E$ but $\lim_{b\to\infty}\gamma_b\simeq 4.15$, rather different from the value of γ_E quoted above. Nevertheless, the FSS results were based on hypothesis which, until now, have not been proved or numerically tested.

Our aim here is to investigate the statistic of SAW's using the series expansion method to analyze the limiting behavior of γ_b as $b\to\infty$. We use a graph counting method that provides an exact analytical recursion relation between the number of embeddings of n-step SAW's in consecutive stages of the iterative process of construction of the fractal lattice [1]. In the limit of infinite iterations, that gives the *exact* density of n-step distinct SAWs $c_n(b)$ (averaged over all possible starting points), for each infinite fractal lattice labeled by b. Contrary to previous findings, to our knowledge we obtain the first numerical results consistent with $\lim_{b\to\infty}\gamma_b=\gamma_E$. We also give analytical arguments supporting this convergence based on the large-n contribution of the series expansion of the chain-generating function near criticality.

II. NUMERICAL RESULTS

We have exactly evaluated $c_n(b)$ in (1) for $n\leqslant n_{\max}$, where $13\leqslant n_{\max}\leqslant 20$ for $2\leqslant b\leqslant 100$. In Table I we present as an illustration $c_n(b)$ for $b=8$, $1\leqslant n\leqslant 16$, and for $b=80$, $1\leqslant n\leqslant 13$.

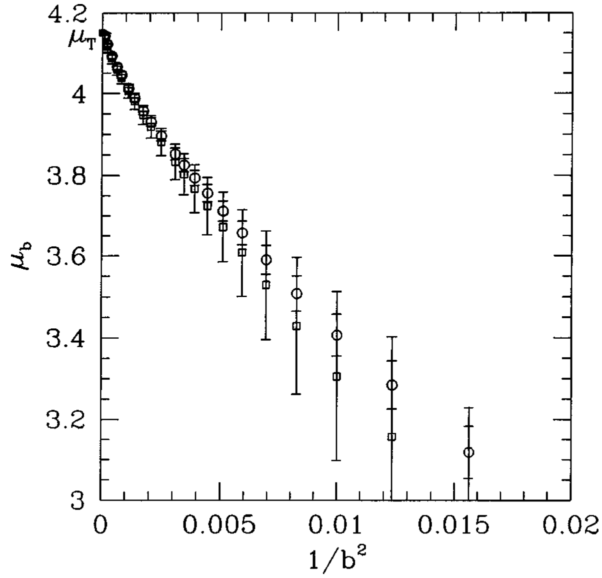

FIG. 2. Plot of μ_b vs $1/b^2$. The results of the present work are denoted by □. We also plot for comparison the results of Ref. [1], denoted by ○. Convergence toward the triangular value μ_T is observed.

The critical behavior of $C_b(x)$ [Eq. (2)] is then analyzed from the $n_{\max}$ order series expansions obtained. This technique was successful in predicting the large-b behavior of μ_b toward μ_T with good accuracy [1]. We present a numerical analysis of $c_n(b)$ which leads to the evaluation of both μ_b and γ_b.

From the series expansion of Eq. (2), $c_n(b)$ may be written as

$$c_n(b)=A_b\mu_b^n n^{\gamma_b-1}F_b(n), \tag{3}$$

where A_b is a constant and $\lim_{n\to\infty}F_b(n)=1$ for each b. Then

$$\ln c_n(b)=R_b+S_bn+T_b\ln n+\varepsilon_b(n), \tag{4}$$

with

$$\begin{aligned} R_b&\equiv\ln A_b,\\ S_b&\equiv\ln\mu_b,\\ T_b&\equiv\gamma_b-1,\\ \varepsilon_b(n)&=\ln F_b(n). \end{aligned} \tag{5}$$

As $\lim_{n\to\infty}\ln F_b(n)=0$, we consider $\varepsilon_b(n)$ as a correction term in Eq. (4). The parameters R_b, S_b, and T_b are chosen by least-square fit in order to minimize the error,

$$\varepsilon=\tfrac{1}{2}\sum_{n=1}^{\infty}[\varepsilon_b(n)]^2. \tag{6}$$

As an illustration, in Fig. 1 we plot $\ln c_n(b)$ versus n for $b=2, 5, 40$, and 100, and, for comparison, we plot the adjusted curve $(R_b+S_bn+T_b\ln n)$ versus n with the best-fit parameters. The agreement is rather good.

In Fig. 2 we plot μ_b versus $1/b^2$. [Previously it has been

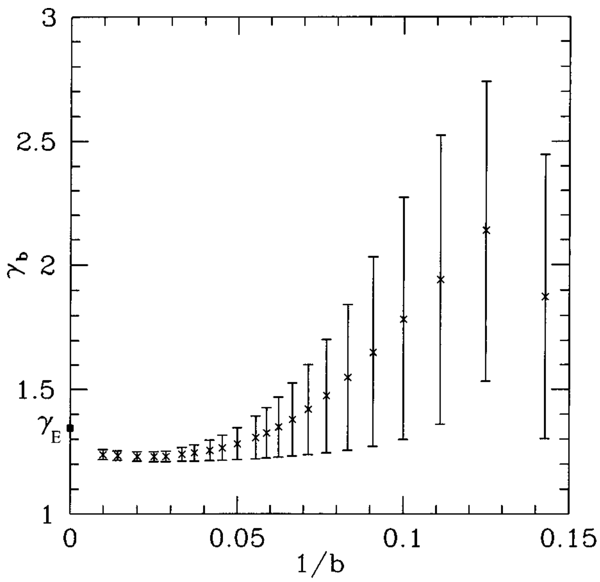

FIG. 3. Plot of γ_b vs $1/b$. We also plot the Euclidean value γ_E.

shown that the first correction term $(1/b)$ is null [1]). The error bars were obtained from the standard deviation of the best-fit parameters S_b when one considers different sets of n values to find the adjusted curve. This plot shows the convergence of μ_b toward μ_T when b is sufficiently large. The decrease of error bars as b increases signs that the series expansions become well behaved (the convergence is faster) as the underlying fractal lattices approaches the uniform Euclidean lattice.

In Fig. 3, we plot γ_b versus $1/b$. The error bars were obtained analogously from the standard deviation of the best-fit parameters T_b. For $b \leqslant 8$ the numerical estimates include the exact results [4], and for $b > 20$ the γ_b values are smaller than γ_E, deviating at most 10% from γ_E. From this plot, it is not possible to ensure the convergence of the numerical estimates of γ_b toward the Euclidean value γ_E as $b \to \infty$, but, given that the deviations are small, one can expect that the convergence to γ_E from below actually occurs. The convergence of the connective constant of the GSG family (a non-universal parameter) settled by the series expansion technique gives further support to this conjecture.

III. ANALYTIC RESULTS FOR THE EUCLIDEAN LIMIT

We now present a theoretical argument supporting that asymptotically γ_b should tend to γ_E. Consider Eq. (4) and the vectors (note that only $\vec{Z}_b$ depends on b)

$$\begin{aligned}
\vec{x} &\equiv (1,2,\ldots,n,\ldots),\\
\vec{y} &\equiv (\ln 1, \ln 2, \ldots, \ln n, \ldots),\\
\vec{Z}_b &\equiv (\ln c_1(b), \ln c_2(b), \ldots, \ln c_n(b), \ldots),\\
\vec{1} &\equiv (1,1,\ldots,1,\ldots).
\end{aligned} \tag{7}$$

Error (6) is $\varepsilon = \frac{1}{2} \| \vec{Z}_b - R_b \vec{1} - S_b \vec{x} - T_b \vec{y} \|^2$. To minimize ε, one should solve the system

$$R_b \vec{1} \cdot \vec{1} + S_b \vec{x} \cdot \vec{1} + T_b \vec{y} \cdot \vec{1} = \vec{Z}_b \cdot \vec{1},$$

$$R_b \vec{1} \cdot \vec{x} + S_b \vec{x} \cdot \vec{x} + T_b \vec{y} \cdot \vec{x} = \vec{Z}_b \cdot \vec{x},$$

$$R_b \vec{1} \cdot \vec{y} + S_b \vec{x} \cdot \vec{y} + T_b \vec{y} \cdot \vec{y} = \vec{Z}_b \cdot \vec{y},$$

or, in matrix notation, $U X_b = Y_b$, with

$$X_b \equiv \begin{pmatrix} R_b \\ S_b \\ T_b \end{pmatrix}, \quad Y_b \equiv \begin{pmatrix} \vec{Z}_b \cdot \vec{1} \\ \vec{Z}_b \cdot \vec{1} \\ \vec{Z}_b \cdot \vec{1} \end{pmatrix},$$
$$U \equiv \begin{pmatrix} \vec{1}\cdot\vec{1} & \vec{x}\cdot\vec{1} & \vec{y}\cdot\vec{1} \\ \vec{1}\cdot\vec{x} & \vec{x}\cdot\vec{x} & \vec{y}\cdot\vec{x} \\ \vec{1}\cdot\vec{y} & \vec{x}\cdot\vec{y} & \vec{y}\cdot\vec{y} \end{pmatrix}. \tag{8}$$

As $\vec{1}$, $\vec{x}$, and $\vec{y}$ are linearly independent, U is invertible:

$$X_b = U^{-1} Y_b. \tag{9}$$

The critical properties of SAW's are obtained when the number of steps $n \to \infty$. The existence of X_b in this limit can be settled from the following arguments. Considering a cutoff N, one can rewrite U as

$$\begin{pmatrix} \sum_{n=1}^{N} 1 & \sum_{n=1}^{N} n & \sum_{n=1}^{N} \ln n \\ \sum_{n=1}^{N} n & \sum_{n=1}^{N} n^2 & \sum_{n=1}^{N} n \ln n \\ \sum_{n=1}^{N} \ln n & \sum_{n=1}^{N} n \ln n & \sum_{n=1}^{N} (\ln n)^2 \end{pmatrix}.$$

The largest element of U is $\Sigma_{n=1}^{N} n^2 \sim N^3$. Then

$$\|U\| \geqslant N^3 \to \|U^{-1}\| \leqslant N^{-3}. \tag{10}$$

On the other hand, from Eq. (8),

$$\begin{aligned}
\|Y_b\|^2 &= (\vec{Z}_b \cdot 1)^2 + (\vec{Z}_b \cdot \vec{x})^2 + (\vec{Z}_b \cdot \vec{y})^2 \\
&= \left(\sum_{n=1}^{N} \ln c_n(b) \right)^2 + \left(\sum_{n=1}^{N} n \ln c_n(b) \right)^2 \\
&\quad + \left(\sum_{n=1}^{N} \ln n \ln c_n(b) \right)^2.
\end{aligned} \tag{11}$$

From the largest contribution in Eq. (11),

$$\|Y_b\| \sim \sum_{n=1}^{N} n \ln c_n(b) \sim N^3, \tag{12}$$

where from Eq. (4), we have used that $\ln c_n(b) \sim n$ for large n.

From Eqs. (9), (10), and (12), we conclude that X_b is finite in the thermodynamic limit $N \to \infty$. Once Eq. (9) is well defined in the critical regime for each fractal labeled by b, we perform the Euclidean limit

$$\lim_{b\to\infty} X_b = \lim_{b\to\infty} U^{-1} Y_b = U^{-1} \lim_{b\to\infty} Y_b , \qquad (13)$$

where U is b independent.

In Ref. [1], it was obtained that $\lim_{b\to\infty} c_n(b) = c_n(T)$, the density of SAW's of the underlying triangular lattice. Using Eqs. (7) and (8), then $\lim_{b\to\infty} Y_b = Y_T$, the corresponding value of the triangular lattice. Finally, from Eq. (13),

$$\lim_{b\to\infty} X_b = U^{-1} Y_T = X_T , \qquad (14)$$

with X_T given by Eq. (8) with triangular parameters R_T, S_T, and T_T.

Consequently, from Eqs. (5) and (14),

$$\lim_{b\to\infty} \mu_b = \mu_T \quad \text{and} \quad \lim_{b\to\infty} \gamma_b = \gamma_E .$$

IV. DISCUSSION

The convergenge of critical exponents on fractals to those on uniform integer dimensionsal lattices is quite subtle, as explained in the text. In this work, we present results for the connective constant μ_b and for the critical exponent γ_b of SAW's on a family of fractals that approaches the triangular lattice asymptotically as $b\to\infty$.

The SAW statistic is evaluated directly on the GSG family. Previous results in the literature regarding critical exponents for these lattices were obtained from other fractal families which are supposed to belong to the same universality class. For this reason they were not be able to provide estimates for μ_b which is a nonuniversal parameter.

We present analytic arguments supporting the convergence $\lim_{b\to\infty}\mu_b = \mu_T$ and $\lim_{b\to\infty}\gamma_b = \gamma_E$. Our numerical estimates for μ_b clearly shows that $\lim_{b\to\infty}\mu_b = \mu_T$, the connective constant of the underlying triangular lattice. From this, one can expect the analogous numerical convergence of the critical exponent.

Although the numerical estimates of γ_b were obtained for a large range of b ($b \leqslant 100$), it was not sufficient to establish the numerical convergence of γ_b toward γ_E. That means that we have not reached the asymptotic regime, which would occur for larger b. Nevertheless, our values deviate by at most 10% from γ_E, displaying a behavior in accordance with our analytic prediction.

In fact, to our knowledge this is the first numerical result showing this trend. Previous numerical findings[2], based on MCRG simulations of 5×10^6 walks for $b\leqslant 80$, provide γ_b estimates that departs from γ_E as b increases. However, multiplying the number of sites of the fractal generator $[N_S = (b+1)(b+2)/2]$—where the simulations were performed by our exact results for the density of n-step SAW's $c_n(b)$, shown in Table I, one finds that the number of n-step SAW's grows as $b^2 c_n(b)$ which is very large compared with the number of Monte Carlo realizations, especially for large b. This could explain the disagreement between the MCGR results and the present work for large b.

From the MCGR data is not possible to obtain any limiting value for γ_b when $b\to\infty$. Although the authors argue that their results would be consistent with the limiting value $\gamma_{FS} \simeq 4.15$ provided by a FSS hypothesis [3], the largest value found in the simulations was $\gamma \simeq 2.2$, which means a 50% deviate from the conjectured asymptotic value. In addition, the MCRG results for ν_b seems inconsistent with the limiting value $\nu_{FS} = \nu_E$ provided by the same FSS hypothesis. These controversial results call for additional studies.

Our numerical estimates of μ_b and γ_b rely upon series expansions that are exact order by order. Each term of order n is obtained from an exact counting of $c_n(b)$ for each infinite fractal lattice labeled by b. This means that the SAW statistics is calculated taking into account the existence of lacunas of all length scales, capturing the full geometry, in contrast with MC results that suffer from finite-size effects.

Finally, the series expansion method gives the most reliable results for Euclidean lattices. This gives confidence in the results based on the extension of this method for fractals.

The finite-size scaling predictions are based on a hypothesis regarding the dependence of critical quantities on b that has not been proved so far for the fractals families studied here. On the other hand, our analytic arguments are performed on a firm mathematical basis.

The results presented here leads to the conclusion that the critical behavior of SAW's on the SGS family of fractals shows a uniform convergence to the Euclidean behavior as $b\to\infty$.

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